Math 263 - Final - Solutions

1. $y' + 2\frac{\cos(x)}{\sin(x)} y = 4\frac{\cos(x)}{\sin(x)}$.

Integrating factor $= \exp\left(\int 2\frac{\cos(x)}{\sin(x)}\,dx\right)$

$= \sin^2(x)$.

$\Rightarrow \left(\sin^2(x)\, y\right)' = 4\cos(x)\sin(x)$

$= 2\sin(2x)$

$\Rightarrow \sin^2(x)\, y = -\cos(2x) + C$.

$y = \frac{-\cos(2x) + C}{\sin^2(x)}$.

I.C: $0 = \frac{-\cos(\pi) + C}{\sin^2(\frac{\pi}{2})}$.

$= \frac{-(-1) + C}{1} \Rightarrow C = -1$.

$$\boxed{y = \frac{-1 - \cos(2x)}{\sin^2(x)}. \quad \text{valid on } (0, \pi).}$$

2. Exact? Try

$M_y - N_x = 3y^2 - (2y^2 - x^3 y^{-1})$.

$= (y^3 + x^3)/y. \neq 0$

Try y-integrating factor:

$\frac{M_y - N_x}{-M} = -\frac{1}{y}$ ✓ $\Rightarrow$

Integrating factor

$\mu = e^{\int -\frac{1}{y}} = \frac{1}{y}$.

equation becomes:

$$\left(\frac{x^3}{y} + y^2\right)dx + \left(2xy - \frac{x^4}{4y^2} + 2y^2\right) = 0$$

exact, giving $\boxed{\left(\frac{x^4}{4y} + xy^2 + \frac{2}{3}y^3\right) = C}$

3. $y'' - 4y' + 5y = 2\sin(x)$, $y(0) = y'(0) = 0$

- Homogeneous: $\lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\Rightarrow$ sols $y_1 = e^{2x}\cos(x)$

$y_2 = e^{2x}\sin(x)$.

- Particular sol'n:

$y_p = A\sin(x) + B\cos(x)$

$y_p' = +A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x)$

$y_p'' - 4y_p' + 5y_p = (-A + 4B + 5A)\sin(2x) + (-B - 4A + 5B)\cos(2x) = 2\sin(x)$.

$\Rightarrow \begin{cases} -4A + 4B = 0 \\ 4A + 4B = 2. \end{cases} \Rightarrow \begin{matrix} A = B \\ 8A = 2 \end{matrix}$ $\Rightarrow A = B = \frac{1}{4}$

$y_p = \frac{1}{4}(\sin(x) + \cos(x))$

- Initial cond'n:

$y = c_1 e^{2x}\cos(x) + c_2 e^{2x}\sin(x) + \frac{1}{4}(\sin(x) + \cos(x))$

$y' = c_1(2e^{2x}\cos(x) - e^{2x}\sin(x)) + c_2(2e^{2x}\sin(x) + e^{2x}\cos(x)) + \frac{1}{4}(\cos(x) - \sin(x))$

$y(0) = c_1 + \frac{1}{4} = 0 \quad c_1 = -\frac{1}{4}$

$y'(0) = 2c_1 + c_2 + \frac{1}{4} = 0 \quad c_2 = \frac{1}{4}$

$$y = \frac{-1}{4}e^{2x}\cos(x) + \frac{1}{4}e^{2x}\sin(x) + \frac{1}{4}\cos(x) + \frac{1}{4}\sin(x).$$

4) Solve $x^2 y'' + x y' - y = 2x^2 \cos(x)$.

- Homog: $y = x^\alpha$ get $\alpha(\alpha-1) + \alpha - 1 = 0$

$\alpha^2 - 1 = 0$ $\alpha = \pm 1$.

$y_1 = x$, $y_2 = x^{-1}$

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1}.$$

- Solve $y'' + \frac{y'}{x} - \frac{y}{x^2} = 2\cos(x)$

by var'n of parameters:

$y_p = v_1 y_1 + v_2 y_2$, with

$$v_1 = \int \frac{-y_2 \cdot 2\cos(x)}{-2x^{-1}} = \int \cos(x).$$

$$= \sin x.$$

$$v_2 = \int \frac{y_1 (2\cos(x))}{-2x^{-1}} = \int -x^2 \cos(x).$$

$$= (-x^2+2)\sin(x) - 2x\cos(x)$$

(by parts, twice.

$$\Rightarrow \boxed{y = x(c_1 + \sin(x)) + x^{-1}(c_2 + (-x^2+2)\sin(x) - 2x\cos(x))}$$

$$= c_1 x + c_2 x^{-1} + 2x^{-1}\sin(x) - 2\cos(x)$$

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5) $y^{(iv)} + 2y''' + y'' = 1 + 9e^{2x}$.

Homog: $\lambda^4 + 2\lambda^3 + \lambda^2 = (\lambda+1)^2 - \lambda^2$.

$\Rightarrow$ roots $\lambda = -1$ (twice)

$\lambda = 0$ (twice)

$\Rightarrow$ sols $y_1 = e^{-x}$, $y_2 = x e^{-x}$

$y_3 = 1$, $y_4 = x$.

5/ particular solutions

a) $y^{(iv)} + 2y''' + y'' = 1$:

1, x solve homog. $\rightarrow$ try $y = Ax^2$.

get $A = \frac{1}{2}$ $\quad y_{p_1} = \frac{x^2}{2}$.

b) $y^{(iv)} + 2y''' + y'' = 9e^{2x}$.

Try $y = Ae^{2x}$.

$\Rightarrow (16 + 2\cdot 8 + 4) A e^{2x} = 9e^{2x}$.

$\Rightarrow 36A e^{2x} = 9e^{2x}$

$A = \frac{1}{4}$

$y_{p_2} = \frac{e^{2x}}{4}$

$\Rightarrow$ $\boxed{y = c_1 e^{-x} + c_2 x e^{-x} + c_3 + c_4 x + \frac{x^2}{2} + \frac{e^{2x}}{4}}$.

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6) $y'' + 2y = \delta(t-\pi)$ $\quad y(0)=1,\ y'(0)=2$.

Take $\mathcal{L}$ of this:

$(s^2 \mathcal{L}\{y\} - s - 2) + 2\mathcal{L}\{y\} = e^{-\pi s}$.

$\Rightarrow \mathcal{L}\{y\} = \frac{1}{s^2+2}\left(s + 2 + e^{-\pi s}\right)$

$= \frac{s}{s^2+2} + \sqrt{2}\left(\frac{\sqrt{2}}{s^2+2}\right) + \frac{e^{-\pi s}}{s^2+2}$

$\Rightarrow$ $\boxed{y = \cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) + \frac{1}{\sqrt{2}}\left(u(t-\pi)\cdot\sin(\sqrt{2}(t-\pi))\right)}$

7 | a) Row space : $\begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$

Col. space : $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$.

b) Kernel: $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

c) Apply Gram-Schmidt:

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{(-1,3,1)\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}{(2,2,1)\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} - \frac{5}{9} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -19 \\ 17 \\ 4 \end{pmatrix}$$

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8) Symmetric matrix $\Rightarrow$ real eigenvalues

- $\det \begin{pmatrix} 2-\lambda & 4 \\ 4 & -4-\lambda \end{pmatrix} = \lambda^2 + 2\lambda - 24 = (\lambda+6)(\lambda-4) = 0$

$\Rightarrow$ eigenvalues $-6, 4$.

- eigenvectors:

$\lambda = -6$: $\begin{pmatrix} 2-(-6) & 4 \\ 4 & -4-(-6) \end{pmatrix} v = 0 \Rightarrow v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = 4$ $\begin{pmatrix} 2-4 & 4 \\ 4 & -4-4 \end{pmatrix} v = 0 \Rightarrow v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\Rightarrow \boxed{D = \begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}}.$$

- projections:

$$\Pi_1 = v_1 \cdot v_1^T = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\Pi_2 = v_2 \cdot v_2^T = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

9- We are told that there is one solution, of the form. $z_1 = e^{5t+2it}\begin{pmatrix}1\\1-2i\end{pmatrix}$. This implies that its complex conjugate $z_2 = e^{5t-2it}\begin{pmatrix}1\\1+2i\end{pmatrix}$ is also a solution. Taking real and imaginary parts, we have:

$$y_1 = \frac{z_1+z_2}{2} = e^{5t}\begin{pmatrix}\frac{e^{2it}+e^{-2it}}{2}\\ \frac{e^{2it}+e^{-2it}}{2} + -2i\,\frac{e^{2it}-e^{-2it}}{2}\end{pmatrix}$$

$$= e^{5t}\begin{pmatrix}\cos(2t)\\ \cos(2t) + 2\sin(2t)\end{pmatrix}$$

$$y_2 = \frac{z_1-z_2}{2i} = e^{5t}\begin{pmatrix}\frac{e^{2it}-e^{-2it}}{2i}\\ \frac{e^{2it}-e^{-2it}}{2i} - \frac{e^{it}+e^{-2it}}{2}\end{pmatrix}$$

$$= e^{5t}\begin{pmatrix}\sin(2t)\\ \sin(2t) - 2\cos(2t)\end{pmatrix}$$

At $t=0$, $y_1 = \begin{pmatrix}1\\1\end{pmatrix}$, $y_2 = \begin{pmatrix}0\\-2\end{pmatrix}$. $\Rightarrow y_1 + \frac{1}{2}y_2 = \begin{pmatrix}1\\0\end{pmatrix}$.

$\Rightarrow$ solution $y_1 + \frac{1}{2}y_2 = e^{5t}\begin{pmatrix}\cos(2t)+\frac{1}{2}\sin(2t)\\ \frac{5}{2}\sin(2t)\end{pmatrix}$

these all tend to zero as one goes to $(+\infty)$